Introductory note to *1951*

1. Historical information and overview

On 26 December 1951, at a meeting of the American Mathematical Society at Brown University, Gödel delivered the twenty-fifth Josiah Willard Gibbs Lecture, "Some basic theorems on the foundations of mathematics and their implications". It is not known when he received the invitation to give this lecture. It is probable, as Wang suggests (*1987*, pages 117–18), that the lecture was the main project Gödel worked on in the fall of 1951. In letters to Rita Dickstein (21 March 1953) and Yehoshua Bar-Hillel (7 January 1954), preserved in Gödel's *Nachlass*, he expressed his intention to publish the lecture in the *Bulletin of the American Mathematical Society.* These letters lend some support to the conjecture that he continued to work on the text after 1951. The lecture was included on a list Gödel made up bearing the title "Was ich publizieren könnte" (see the preface to this volume) and also preserved in the *Nachlass*. No correspondence with the editors of the *Bulletin* is known, however, and the only text we have is handwritten (and of a rather intricate structure; see the textual notes). Since other papers of Gödel survive in typescripts—in the cases of **1946/9* and **1953/9* in several versions—it may also be conjectured that he did not come close to sending it off for publication.

Gödel's lecture may be divided into two parts, the first of which is an exposition of certain logical results and of philosophical views that he regards as direct consequences of those results. In this part of the lecture Gödel tries to establish that the results show mathematics to be "incompletable" or "inexhaustible", and that one of them demonstrates that "*either ... the human mind* (*even within the realm of pure mathematics*) *infinitely surpasses the powers of any finite machine, or else there exist absolutely unsolvable diophantine problems*" (page 13). (It will be explained below what Gödel understands by a "diophantine problem".) By an "absolutely undecidable" problem, Gödel means one that is undecidable, "not just within some particular axiomatic system, but by *any* mathematical proof the human mind can conceive" (page 13).

In the second, more avowedly philosophical, part of the lecture, Gödel's main concern is to adduce a number of considerations favoring the standpoint called realism or Platonism, which can be defined, in Gödel's own words, as the view that mathematical objects and "concepts form an objective reality of their own, which we cannot create or change, but only perceive and describe" (page 30).

2. Set theory and the incompletability of mathematics

The attempt to axiomatize set theory is the first of two illustrations Gödel provides of what he means by the inexhaustibility of mathematics. Gödel claims that in order to avoid the paradoxes "without bringing in something entirely extraneous[a] to actual mathematical procedure, the concept of set must be axiomatized in a stepwise manner" (page 3). He then proceeds to lay out the "iterative" or "cumulative" hierarchy of sets: we begin with the integers and iterate the power-set operation through the finite ordinals. This iteration is an instance of a general procedure for obtaining sets from a set A and well-ordering R: starting with A, iterate the power-set operation through all ordinals less than the order type of R (taking unions at limit ordinals). Specializing R to a well-ordering of A (perhaps one whose ordinal is the cardinality of A) yields a new operation whose value at any set A is the set of all sets obtained from A at some stage of this procedure, a set far larger than the power set of A. We can require that this new operation, and indeed *any* set-theoretic operation, can be so iterated, and that there should also always exist a set closed under our iterative procedure when applied to any such operation.

Axioms can be formulated to describe the sets formed at various stages of this process. But as there is no end to the sequence of operations to which this iterative procedure can be applied, there is none to the formation of axioms. "... nor can there ever be an end to *this* procedure of forming the axioms, because the very formulation of the axioms up to a certain stage gives rise to the next axiom" (page 5).

The elaboration of Gödel's views on the iterative concept of set found in *Wang 1974* makes it clear that the axioms we thus formulate will imply all those of ZF, including the axioms of replacement. An interesting conclusion is immediate: on Gödel's view, the iterative concept of set is only *partially* embodied in the theory ZF.

Gödel seems never to have wavered from the view that ZF only partially characterizes the concept of set. In **1933o* he speaks of "... an infinity of systems, and whichever system you choose out of this infinity, there is one more comprehensive, i.e., one whose axioms are stronger" (page 10). And as late as *1964*, footnote 20, he states that Mahlo's axioms, which assert the existence of Mahlo cardinals but which cannot be proved in ZF, are "implied by the general concept of set".

Gödel observes that higher-level set-theoretic axioms will entail the solution of certain diophantine problems of level 0 left undecided by the

[a]It is conceivable that he may have had in mind Quine's set theories NF and ML, in which whether a formula counts as an axiom depends on whether it satisfies a somewhat artificial syntactical restriction.

preceding axioms; the problems, moreover, take a particularly simple form, viz., to determine the truth or falsity of sentences $\forall \mathbf{x} \exists \mathbf{y} P(\mathbf{x}, \mathbf{y}) = 0$, where $\mathbf{x}$ and $\mathbf{y}$ are sequences of integer variables and $P(\mathbf{x}, \mathbf{y})$ is a polynomial with integer coefficients. Let us call this class of sentences "class A". (For Gödel's proof that undecidable sentences can be taken to be in class A, see **193?*).

3. The incompleteness theorems and incompletability

Not surprisingly, Gödel's own incompleteness theorems provide his second illustration of the incompletability of mathematics. Invoking the notion of a Turing machine, he states that the first theorem "is equivalent to the fact that there exists no finite procedure for the systematic decision of all diophantine problems of the type specified" (page 9); little further mention is then made of the first theorem, since it is the *second* theorem (page 10) that he thinks makes the incompletability of mathematics particularly evident:

> *For any well-defined system of axioms and rules ... the proposition stating their consistency (or rather the equivalent number-theoretical proposition) is undemonstrable from these axioms and rules, provided these axioms and rules are consistent and suffice to derive a certain portion of the finitistic arithmetic of integers.*

Gödel's argument that his second theorem shows the incompletability of mathematics runs as follows: No one can set up a formal system and consistently state about it that he perceives (with mathematical certitude) that its axioms and rules are correct and that he believes that they contain all of mathematics, for anyone who claims to perceive the correctness of the axioms and rules must also claim to perceive their consistency; but since the consistency of the axioms is not provable in the system, the person is claiming to perceive the truth of something that cannot be proved in the system, and is therefore obliged to abandon the claim that the system contains all of mathematics.

Gödel moves immediately to prevent a possible misunderstanding. He distinguishes the system of all true mathematical propositions from that of all demonstrable mathematical propositions, calling these mathematics in the objective and subjective senses, respectively, and claims that it is only objective mathematics that no axiom system can fully comprise. He adds that we could not, however, know of any finite rule that might happen to produce all of subjective mathematics that it is correct. The ground for both claims is the indemonstrability of the assertion of consistency. To be sure, we could successively come to recognize, of each

proposition produced by subjective mathematics, that that proposition is correct; but we could not know the general proposition that they are *all* correct.

Were there to be such a rule, Gödel says, the mind would be "equivalent to a finite machine that, however, is unable to understand completely its own functioning" (page 12), again on the ground that the insight that the brain produces only "correct (or only consistent) results would surpass the powers of human reason" (footnote 14). Gödel supposes that if a (consistent) machine "completely understands" its own functioning, then it can recognize its own consistency.

Gödel also holds that if the human mind is "equivalent to a finite machine" (page 12), then there is a finite rule producing all the evident axioms of demonstrable mathematics. Since the assertion of consistency can be recast as a sentence in class A, he takes it that it follows that either the human mind surpasses the powers of a finite machine or there exist simple problems about the natural numbers not decidable by any proof the human mind can conceive. He calls his conclusion a "mathematically established fact" (page 13) that seems to him of great philosophical interest.

There is a gap between the proposition that no finite machine meeting certain weak conditions can print a certain formal sentence (which will depend on the machine) and the statement that if the human mind is a finite machine, there exist truths that cannot be established by any proof the human mind can conceive. It is not that no proposition about the "human mind" or human beings or brains can ever be validly inferred from a mathematical proposition. (On the contrary: since 91 is composite, no human being will ever come to know that 91 is prime.) What may be found problematic in Gödel's judgment that his conclusion is of philosophical interest is that it is certainly not obvious what it means to say that the human mind, or even the mind of some one human being, *is* a finite machine, e.g., a Turing machine. And to say that the mind (at least in its theorem-proving aspect), or *a* mind, may be represented by a Turing machine is to leave entirely open just *how* it is so represented. Nevertheless, the following statement about minds, replete with vagueness though it may be, would indeed seem to be a consequence of the second theorem: If there is a Turing machine whose output is the set of sentences expressing just those propositions that can be proved by a mind capable of understanding all propositions expressed by a sentence in class A, then there is a true proposition expressed by a sentence in class A that cannot be proved by that mind.

Apart from the difficulties involved in deriving from the second incompleteness theorem the disjunctive claim that either the mind is not a finite machine or there exist absolutely undecidable mathematical propositions, a further problem for Gödel's view is that the supposition that

the second alternative holds does not seem particularly surprising or remarkable at present. (Of course, it may well be that the existence of propositions whose truth we could never recognize is unremarkable precisely *because* we have come to understand the incompleteness theorems so well.) Why, we may wonder, should there *not* be mathematical truths that cannot be given any proof that human minds can comprehend? It may be noted that there are many persons who, influenced by the picture of the mind as a Turing machine, find the falsity of the first and the truth of the second alternative a pair of propositions they are quite willing to maintain. Others, while reserving judgment on the question whether (the mathematical abilities of) a mind can be (represented by) a Turing machine, simply find it extremely plausible that there are mathematical truths unprovable by any humanly comprehensible proof.[b]

According to Wang (*1974*, pages 324–326), Gödel believed that Hilbert was right to reject the second alternative. Otherwise, by asking unanswerable questions while asserting that only reason can answer them, reason would be irrational. (This view may derive from Kant's opinion that "there are sciences the very nature of which requires that every question arising within their domain should be completely answerable in terms of what is known, inasmuch as the answer must issue from the same sources from which the question proceeds" [A 476/B 504, translation from *Kant 1933*]. Kant cites pure mathematics as one such science [A 480/B 508].[c]) Not only did Gödel reject the second alternative, he appears to have thought (at least late in his life) that there were independent reasons for accepting the first as well: Remark 3 of *1972a* is an argument against Turing's view that "mental procedures cannot go beyond mechanical procedures" (page 306).[d]

Gödel's disjunctive conclusion concerning the significance of his incompleteness theorems stands in contrast with the conclusion drawn by writers such as Ernest Nagel and James R. Newman (*1958*), J.R. Lucas (*1961*), and Roger Penrose (*1989*) to the effect that the theorems show outright that the mind is not a Turing machine, since, as they suppose, the mind can see with mathematical certainty that any Turing machine that it might be alleged to be (or be represented by) is actually consistent, and can therefore prove a proposition not provable by that machine. The classic reply to these views was given by Hilary Putnam

[b] In their introductory note to Remark 2 of *1972a*, Feferman and Solovay suggest one possible example. Cf. these *Works*, Vol. II, p. 292.

[c] I am grateful to Carl Posy and Sally Sedgwick for calling these passages in the *Critique of pure reason* to my attention.

[d] A critical assessment of Gödel's argumentation is given in unpublished work of Warren Goldfarb.

(*1960*): Merely to find from a given machine M a statement S for which it can be proved that M, if consistent, cannot prove S is not to *prove* S—even if M *is* consistent. It is fair to say that the arguments of these writers have as yet obtained little credence.

Before we turn to the more philosophical part of Gödel's lecture, let us mention some questions that his discussion suggests. Do the impossibility of axiomatizing the concept of set and that of axiomatizing the whole of mathematics bear any interesting relation to each other? Indeed, is there a significant general phenomenon of inexhaustibility or incompleteness of which they are both examples (and if so, what is it)? Is there even a third instance of the incompletability or inexhaustibility of mathematics to be cited?

4. Realism, or Platonism

Gödel remarks that if either the mind is not a finite machine or there exist absolutely undecidable propositions, then the philosophical conclusions to be drawn are "very decidedly opposed to materialistic philosophy" (page 15). If the first alternative holds and the mind's operations cannot be reduced to those of the brain, which is made out of a finite number of neurons and their connections, then vitalism, he states, would seem to be inescapable. Gödel claims that this alternative is not known to be false and that some of the "leading men in brain and nerve physiology" (page 17) deny the possibility of a purely mechanistic explanation of mental processes.

The second alternative, which, he says, "seems to disprove the view that mathematics is only our own creation" (page 15), appears to imply some version of realism or Platonism about the objects of mathematics and gives Gödel considerably more to say.

A creator, he says, "necessarily knows all properties of his creatures, because they can't have any others except those he has given to them" (page 16). Gödel considers poor the objection that the constructor need not know *every* property of what he constructs, that, e.g., we cannot predict the complete behavior of machines we make (or, one might now add, of software we write). His reply to this objection is to argue that if it were correct, it would provide further support for Platonism in mathematics, because we build machines "... out of some given material. If the situation were similar in mathematics, then this material or basis for our constructions ... would force some realistic viewpoint upon us even if certain other ingredients of mathematics were our own creation" (page 18).

Gödel's claim that a creator must know all properties of the things he creates, since they can have no others except those the creator gives

them, may strike the reader as a far-fetched defense of the quite plausible claim that mathematics cannot be only (i.e., entirely) our own creation, at least not if our capacity for proving facts about the natural numbers can be adequately represented by a Turing machine. For how, one might wonder, could it have been *we* who brought about the truth of any true proposition in the absence of a proof of that proposition that we could produce? It might be said that the truth of the proposition is a consequence of stipulations we have made concerning the natural numbers. For this reply to be explanatory, however, "consequence" must mean "deductive consequence" and not (say) "higher-order semantic consequence"; but that is precisely what is *not* the case with regard to an undecidable proposition. In any case, the incompleteness theorems suggest that it is doubtful that the view that mathematics is entirely our own creation can be successfully elaborated. (Gödel does not discuss the objection to the other half of his claim, that objects might in fact acquire properties not bestowed upon them by their creator, for example, as a result of being perceived by others.)

To the objection that the meaning of a proposition about all integers can consist only in the existence of a proof of it, and therefore that neither an undecidable proposition nor its negation is true, Gödel makes a particularly interesting response. He suggests that the abhorrence mathematicians display towards inductive methods in mathematics may be "due to the very prejudice that mathematical objects somehow have no real existence. If mathematics describes an objective world ... there is no reason why inductive methods should not be applied in mathematics" (page 20). Thus his second alternative, that there exist absolutely undecidable propositions, favors the standpoint of empiricism in one respect.

As to what such empirical methods might look like, Gödel offers no concrete suggestion; but, in a footnote, he gives an example of a proposition where probabilities, he says, can be estimated even now: The probability that for each n there is at least one digit $\neq 0$ between the n-th and the n^2-th digits of the decimal expansion of π converges toward 1 as one goes on verifying it for greater and greater n. One may, however, be uncertain whether it makes sense to ask what the probability is of that general statement, given that it has not been falsified below $n = 1{,}000{,}000$, or to ask for which n the probability would exceed .999.

Gödel then gives three arguments supporting the view he calls conceptual realism (or Platonism) and directed against the view that mathematics is our own creation.

According to the first of these, the attainment of great clarity in the foundations of mathematics has helped us little in the solution of mathematical problems; but this, says Gödel, would be impossible were mathematics our "free creation", for then mathematical ignorance

could be due only to failure to understand what we have created (or to computational complexity), and would have to disappear once we attained "perfect clearness".

But, it might be replied, there is no reason to suppose that perfect *clarity* about one of our creations should yield perfect knowledge of it. What is it about creation that guarantees that once we know exactly what a creation of ours is, we must know everything about it? Gödel seems to identify progress in understanding the foundations of mathematics with the attainment of ever greater clarity about mathematics; but, one might think, mathematics might be our own creation and we might have attained perfect clarity about the fundamental properties of what we have created, but we might nevertheless be rather ignorant about non-fundamental properties. There is no reason to suppose that even perfect clarity with respect to all the fundamental properties of our creations must yield *complete* knowledge of those creations.

Gödel's second argument against the view that mathematics is our own creation is that mathematicians cannot create the validity of theorems at will. "If anything like creation exists at all in mathematics, then what any theorem does is exactly to restrict the freedom of creation" (page 22). This consideration is often thought to be a powerful argument on behalf of a realist view of mathematics of the type Gödel wishes to espouse. It is perhaps presented most forcefully as a claim to the effect that the contrary position is confused or incredible: that once it has been made clear exactly *which* objects (including operations, properties and relations) are *in question*, i.e., being talked about, which, all may concede, may well be a matter for choice or decision, the suggestion that there is still room for a decision whether or not those objects have those properties, stand in those relations, etc. cannot be believed to be true. (One might think: Once it is certain that it is 9, 4, 36, multiplication, and equality that are under consideration, how could it possibly be *up to us* whether or not the product of 9 and 4 is 36?) If the creation could not have turned out otherwise, Gödel is arguing, in what sense is there *creation* at all?

Gödel's third argument is that in order to demonstrate certain propositions about the integers, we must employ the concept of a set of integers; but the creation of integers does not "necessitate" that of sets of integers. Thus we appear to be in the "very strange situation indeed" (page 23) of having to make a further creation in order to determine what properties we have given to the integers, which were supposed to be our creation.

This consideration may perhaps best be taken as a "plausibility" argument: Confronted with these facts about integers, sets of integers and our knowledge of the properties of integers, how can we find even slightly plausible the suggestion that mathematics is our own creation?

Whether or not it follows from the view that mathematics is not our own creation that the objects of mathematics have an objective existence that is independent of us will of course depend on how the concepts "objective existence" and "independence" are to be understood: it may be argued that we lack an interpretation of the key terms in this putative consequence under which it is true but not trivially true.

5. Against conventionalism

Conceding that "free creation" is a vague term, Gödel then undertakes to give a more specific refutation of what he takes to be the most precise articulation of that suggestion, the view usually called mathematical conventionalism (though Gödel often refers to it as nominalism), according to which mathematical propositions express only certain aspects of linguistic conventions, "that is, they simply repeat parts of these conventions" (page 23). His discussion is intricate and, in view of the six drafts[e] he made of a projected paper on the philosophy of Rudolf Carnap (at one time the pre-eminent advocate of conventionalism in mathematics), it is highly probable that Gödel was never able to formulate his objections to Carnap's view to his own complete satisfaction. Annotations to the manuscript strongly suggest that he did not intend to read this section of the lecture to his audience in Providence.

He begins by quickly disposing of what he takes to be the simplest form of conventionalism: the view that the truth of mathematical propositions is due solely to the definitions of the terms they contain. Gödel understands this to mean that there is a mechanical method for converting any mathematical truth (and no mathematical falsehood) to an explicit tautology of the form $a = a$ by systematically replacing terms by their definitions. Since any such conversion method would yield a decision procedure for arithmetical truth, this simplest version of conventionalism fails: there is no such decision procedure.

Refined versions, he claims, fare no better. He then attempts to refute the claim that "every demonstrable[f] mathematical proposition can be deduced from the rules about the truth and falsehood of sentences alone (that is, without using or knowing anything else except these rules)" (page 25).

[e]See the introductory note to *1953/9* in this volume.

[f]Although "demonstrable" here might be thought to be a slip for "true," "dem." has been inserted and "true" crossed out at this point in the manuscript. (However, subsequent occurrences are not similarly changed, and the view under attack concerns mathematical truth.)

Gödel's argument is that in order to derive the truth of the axioms of mathematics from rules about the truth and falsity of sentences (as, for example, the truth of $p \vee \sim p$ *is* derivable from the usual rules for truth and falsity of disjunctions and negations), one must apply mathematical and logical concepts and axioms to symbols, sets of symbols, sets of sets of symbols, etc. Thus, one who wants to explain mathematical truth as a species of tautology will find that the explanation cannot proceed without the aid of the axioms of mathematics themselves. Mathematical induction provides the central illustration of Gödel's point: any proof that all instances of mathematical induction are true will appeal, in some way or other, to a form of the principle of mathematical induction itself, or to even stronger set-theoretical principles that cannot plausibly be regarded as rules about the truth and falsity of sentences.

He writes, "while the original idea of this viewpoint was to make the truth of the mathematical axioms understandable by showing that they are tautologies, it ends up with just the opposite, that is, the truth of the axioms must *first* be assumed and *then* it can be shown that, in a suitably chosen language, they are tautologies" (pages 26–7).

By "tautology", it should be noted, Gödel does not mean "truth-functionally valid sentence", but rather something like "sentence whose truth can be deduced from rules stipulating the conditions under which sentences are true and false". Gödel's point is thus that the conventionalists' claim that the truth of true mathematical statements can be deduced from such rules is of no interest if true, since strong mathematical axioms, which can in no way be regarded as "syntactical", will have to be assumed in any valid deduction that shows those statements true.

Gödel argues that any attempt to prove the tautological character of the axioms of mathematics would be a proof of their consistency, which, by his second theorem, cannot be achieved with means weaker than the axioms themselves. It may well be, he notes, that not all of the axioms are needed for the proof of consistency, but it is, he claims, a "practical certainty" that to prove consistency some "abstract concepts", such as "set" or "function of integers", together with the axioms governing these notions, will have to be employed in the proof. Since these notions cannot be considered to be syntactical, it follows, he claims, that syntax cannot rationally warrant our "precritical" beliefs concerning the consistency of classical mathematics.

Although some portions of the theory of abstract concepts can be nominalistically based, and fragments of arithmetic, concerning, e.g., numbers less than 1000, reduced to truth-functionally valid statements, a syntactical justification of mathematical induction is unavailable, "since this axiom itself has to be used in the syntactical considerations" (page 27). Thus the well-known reducibility of arithmetical identities like "$5 + 7 = 12$" to explicit tautologies is misleading, Gödel says, not only

because this statement is contained in a tiny fragment of mathematics whose reducibility to tautology tells us nothing about the rest of mathematics—which includes statements that can be established only with the aid of induction—but also because either "+" is defined so as to refer only to numbers in some finite domain (in which case it does not refer to ordinary addition), or the concept of set, along with axioms about sets, will have to be used in the definitions and proofs.

Gödel then sums up the previous discussion: The essence of the nominalist-conventionalist view is that propositions which we believe express mathematical facts do not do so, and are true simply because of "an idle running of language", i.e., because the rules which determine when propositions are true or false determine that these propositions are true "no matter what the facts are" (page 29). To this view Gödel raises two objections, of which the first summarizes the main point of the foregoing discussion: in any putative proof that mathematics is tautologous or true solely by virtue of some such rules, one would have to use mathematics that is at least as complicated as that being asserted to be tautologous or thus true.

The second objection is that no justification can be given for regarding certain mathematical statements, such as complete induction, as "void of content", for one can easily construct systems in which certain empirical statements are taken as axioms. (For the notion of "content" which Gödel has in mind, see *Carnap 1937*, pages 42 and 120.) As it would clearly be unjustifiable to classify those empirical statements as therefore lacking in content, so, Gödel claims, it would be no more justifiable to regard those mathematical statements as actually *void of content*. Thus, according to Gödel, no ground has been given for thinking that there are no such things as mathematical facts, a claim Gödel calls "the essence of this view" (page 29).

6. Realism and analyticity

Gödel is prepared to acknowledge a grain of truth in the nominalist position. "A mathematical proposition says nothing about the physical or psychical reality existing in space and time, because it is true already owing to the meaning of the terms occurring in it, irrespectively of the world of real things" (page 30). It is an error to think that the meanings of the terms are man-made or that they consist in semantical conventions. Meanings are concepts, which "form an objective reality of their own, which we cannot create or change, but only perceive and describe" (page 30).[g]

[g]Cf. *1944* and Parsons' introductory note thereto.

Philosophers of mathematics and other metaphysicians dispute whether the supposition that mathematical objects or concepts "form an objective reality of their own" is surrogate theology (if not outright craziness), is trivially correct, or is in profound need of philosophical clarification. The matter will not be resolved here. Gödel elaborates, "... a mathematical proposition, although it does not say anything about space-time reality, still may have a very sound objective content, insofar as it says something about relations of concepts" (pages 30–1). But the elaboration helps not at all to settle the dispute. To complicate matters further, it should be noted that the term "world", as in the phrase "world of real things", belongs to the same family as "reality" and "objective", and thus the assertion that mathematical truths say something "about the world" would seem to enjoy the same status (insane, trivial, or unclear) as the claim that they describe an "objective reality".

Gödel's realism takes a strong form: relations between the concepts are not "tautological", because among the axioms that govern the concepts entering into those relations, some must be assumed which are not tautological, but which "follow from the meaning of the primitive terms under consideration" (page 31). Gödel's thought is that statements, such as instances of the comprehension schema in analysis (second-order arithmetic), even those containing quantifiers ranging over all sets of integers, are valid "owing to the meaning of the term 'set'—one might even say they express the very meaning of the term 'set'" (page 32). Gödel distinguishes between truths he calls "analytic" (those true in virtue of the meanings of the terms expressing them or "owing to the nature of the concepts occurring therein") and "tautological" truths (those "devoid of content" or "true owing to our definitions"). It may be emphasized that Gödel does not restrict the term "analytic" to statements of the "oculists are eye-doctors" or "actresses are female" variety. The analytic truths about sets, Gödel states, cannot be proved without appeal to the concept of set itself; and some analytic propositions might well be undecidable, since "our knowledge of the world of concepts may be as limited and incomplete as that of ⟦the⟧ world of things" (page 34). Gödel also discusses the notion of analyticity near the end of *1944*.

Quine's influential attack (*1951*) on the concept of analyticity appeared three months before Gödel delivered his lecture. Gödel's claim that the axioms of set theory are analytic—"true owing to the meanings of the terms they contain or the nature of the concepts those terms express"—is troubling for at least three sorts of reasons that do not entirely depend on Quine's claim that the phrase "true by virtue of meanings" has not been shown to isolate a significant class of truths.

In the first place, there is a difficulty about the truth of the axioms: a number of thoughtful writers believe that the axioms of set theory do not describe anything real, despite Gödel's later assertion (*1964*,

page 271) that they force themselves upon us as being true. It is certainly a sensible view to hold both that Cantor's theory of transfinite numbers is a fantasy and that the standard theorems of elementary number theory and analysis are unquestionably true. In any case, the axioms of set theory lack the kind of obviousness one would have expected *axioms* characterized as "analytic" to enjoy.

Secondly, the axiom of extensionality would seem to be the only axiom of ZF that can be properly said to be true in virtue of the *meaning* of the word "set"; indeed, the axiom is often justified on the ground that the criterion of identity of sets it gives, viz., having the same members, is just part of what is meant by "set" (as opposed, say, to "property") and it is the only one that can be thus defended by an appeal to what "set" means. But since Gödel understands "true in virtue of meanings" as so much wider than "true owing to definitions" that it encompasses all axioms of set theory, Quine's questions re-arise: How is the notion of meaning that Gödel is using to be understood? When the axioms of set theory are said to be true in virtue of the meanings of their constituent terms, what more is said beyond that they are true? What is it for them to be true *in virtue of* the meanings of the terms they contain? A possible rejoinder to the effect that it is not the meaning of "set" but the nature of the concept of set that is of primary importance for Gödel is open to the reply that the last two questions remain unanswered under the replacement of "meanings of terms" by "natures of concepts".

Gödel's view raises worrisome questions of a third sort, suggested in part by later writing (*1964*) of his own: Could not the axioms of set theory be true, not in virtue of the concept of set or the meaning of "set", but simply because sets just happen to be as the axioms have it? Why, one might ask, must our knowledge of sets be mediated solely through our understanding of the *concept* of set; could we not know how matters stand with sets by "something like a perception" of them—to quote from the supplement to *1964*—that is as direct as our perception of the *concept* of set? Even lacking such a perception, might we not acquire quasi-empirical evidence, of a sort that Gödel himself has acknowledged may exist, that certain set-theoretic matters happen to stand one way rather than another? One wonders why a *conceptual* realism should be found any more plausible than an "objectual" realism.[h]

Since "our knowledge of the world of concepts may be as limited and incomplete as that of ⟦the⟧ world of things", Gödel holds that the paradoxes of set theory pose no more threat to his Platonism than the illusion of the stick in water poses to the view that there is an "outer world". The interesting implied suggestion is that we are taken in by

[h] *Parsons 199?* contains further discussion of Gödel's use of the notion of analyticity.

something like an optical *illusion* when we accept the principles that lead to set-theoretic contradiction; perhaps we ought to wonder what we might learn about our mental faculties from a study of these principles.

7. Conclusion

Gödel concludes by claiming that although he has disproved the nominalist standpoint and adduced strong arguments against the more general view that mathematics is our own creation, he could not claim to have proved the realist viewpoint he favors, for to do so would require a survey of the alternatives, a proof that the survey was exhaustive, and a refutation of all the alternatives except realism. Among the alternatives to be refuted are Aristotelian realism, which he characterizes as the view that concepts are aspects or parts of things, and psychologism, which holds that mathematics is nothing but the psychological laws by which thoughts, presumably concerning calculation, etc., occur in us. About Aristotelian realism, Gödel says only that he does not think it tenable. His principal charge against psychologism, reminiscent of Frege's objections, is briefly given: if psychologism were correct, there would be no *mathematical* knowledge, but only knowledge that our mind is so constituted as to consider certain statements of mathematics true. His discussion is admittedly cursory, however, and Gödel gives psychologism, whatever its merits, much less attention than nominalism.

The suggestion with which he closes the lecture may seem utterly strange: that *after sufficient clarification* of the concepts in question, it will be possible to conduct the discussion of these matters "with mathematical rigor", at which time the result will be that the Platonistic view is the only one tenable. (Here he characterizes the position somewhat differently, as "the view that mathematics describes a non-sensual reality, which exists independently both of the acts and ⟦of⟧ the dispositions of the human mind and is only perceived, and probably perceived very incompletely, by the human mind" [page 38].) What is surprising here is not the commitment to Platonism, but the suggestion, which recalls Leibniz's project for a universal characteristic,[i] that there could be a mathematically rigorous discussion of these matters, of which the correctness of any such view could be a "result". Gödel calls Platonism rather unpopular among mathematicians; it is probably rather more popular among them now, forty years after he gave his lecture, in some

[i]In his introductory note to *1944*, Parsons calls Gödel's view, given in the last paragraph of *1944*, that Leibniz did not regard the *Characteristica universalis* as a utopian project "one of his most striking and enigmatic utterances".

measure because of his advocacy of it, but perhaps more importantly because every other leading view seems to suffer from serious mathematical or philosophical defects. Gödel's idea that we shall one day achieve sufficient clarity about the concepts involved in *philosophical* discussion of mathematics to be able to prove, mathematically, the truth of some position in the philosophy of mathematics, however, appears significantly less credible at present than his Platonism.

George Boolos[j]

The translation of the quotation from Hermite at the end of **1951* is by Solomon Feferman, with the assistance of Marguerite Frank.

[j]I am grateful to Cheryl Dawson, John Dawson, Solomon Feferman, Warren Goldfarb, and Charles Parsons for much editorial and philosophical advice.

Some basic theorems on the foundations of mathematics and their implications (**1951*)

Research in the foundations of mathematics during the past few decades has produced some results which seem to me of interest, not only in themselves, but also with regard to their implications for the traditional philosophical problems about the nature of mathematics. The results themselves, I believe, are fairly widely known, but nevertheless I think it will be useful to present them in outline once again, especially in view of the fact that, due to the work of various mathematicians, they have taken on a much more satisfactory form than they had had originally. The greatest improvement was made possible through the precise definition of the concept of finite procedure,[1] which plays a decisive role in these results. There are several different ways of arriving at such a definition, which, however, all lead to exactly the same concept. The most satisfactory way, in my opinion, is that of reducing the concept of finite procedure to that of a

[1]This concept, for the applications to be considered in this lecture, is equivalent to the concept of a "computable function of integers" (that is, one whose definition makes it possible actually to compute $f(n)$ for each integer n). The procedures to be considered do not operate on integers but on formulas, but because of the enumeration of the formulas in question, they can always be reduced to procedures operating on integers.

machine with a finite number of parts, as has been done by the British mathematician Turing. | As to the philosophical consequences of the re- 1
sults under consideration, I don't think they have ever been adequately discussed, or ⟦have⟧ only ⟦just been⟧ taken notice of.

The metamathematical results I have in mind are all centered around, or, one may even say, are only different aspects of one basic fact, which might be called the incompletability or inexhaustibility of mathematics. This fact is encountered in its simplest form when the axiomatic method is applied, not to some hypothetico-deductive system such as geometry (where the mathematician can assert only the conditional truth of the theorems), but to mathematics proper, that is, to the body of those mathematical propositions which hold in an absolute sense, without any further hypothesis. There must exist propositions of this kind, because otherwise there could not exist any hypothetical theorems | either. For example, *some* implica- 2
tions of the form:

> If such and such axioms are assumed, then such and such a theorem holds,

must necessarily be true in an absolute sense. Similarly, any theorem of finitistic number theory, such as $2 + 2 = 4$, is, no doubt, of this kind. Of course, the task of axiomatizing mathematics proper differs from the usual conception of axiomatics insofar as the axioms are not arbitrary, but must be correct mathematical propositions, and moreover, evident without proof. There is no escaping the necessity of assuming some axioms or rules of inference as evident without proof, because the proofs must have some starting point. However, there are widely divergent views as to the extension of mathematics proper, as I defined it. The intuitionists and finitists, for example, reject some of its axioms and concepts, which others acknowledge, such as the law of excluded middle or the general concept of set.

The phenomenon of the inexhaustibility of mathematics,[2] however, always is present in some form, no matter what standpoint is taken. So I might as well explain it for the simplest and most natural standpoint, which takes mathematics as it is, without curtailing it by any criticism. From this standpoint all of mathematics is reducible to abstract set theory. For example, the statement that the axioms of projective geometry imply a certain theorem means that if a set M of elements called points and a set N of subsets of M called straight lines satisfy the axioms, then the theorem

[2]The term "mathematics", here and in the sequel, is always supposed to mean "mathematics proper" (which of course includes formal logic as far as it is acknowledged to be correct by the particular standpoint taken).

3 holds for N, M. Or, to mention | another example, a theorem of number theory can be interpreted to be an assertion about finite sets. So the problem at stake is that of axiomatizing set theory. Now, if one attacks this problem, the result is quite different from what one would have expected. Instead of ending up with a finite number of axioms, as in geometry, one is faced with an infinite series of axioms, which can be extended further and further, without any end being visible and, apparently, without any possibility of comprising all these axioms in a finite rule producing them.[3] This comes about through the circumstance that, if one wants to avoid the paradoxes of set theory without bringing in something entirely extraneous to actual mathematical procedure, the concept of set must be axiomatized in a stepwise manner.[4] If, for example, we begin with the integers, that is, the finite sets of a special kind, we have at first the sets of integers and the axioms referring to them (axioms of the first level), then the sets of sets of integers with their axioms (axioms of the second level), and so on for any finite iteration of the operation "set of".[5] Next we have the set of all
4 these | sets of finite order. But now we can deal with this set in exactly the same manner as we dealt with the set of integers before, that is, consider the subsets of it (that is, the sets of order ω) and formulate axioms about their existence. Evidently this procedure can be iterated beyond ω, in fact up to any transfinite ordinal number. So it may be required as the next axiom that the iteration is possible for *any* ordinal, that is, for any order type belonging to some well-ordered set. But are we at an end now? By no means. For we have now a new operation of forming sets, namely, forming a set out of some initial set A and some well-ordered set B by applying the operation "set of" to A as many times as the well-ordered set B indicates.[6] And, setting B equal to some well-ordering of A, now we can iterate this new operation, and again iterate it into the transfinite. This will give rise to a new operation again, which we can treat in the same way, and so on. So the next step will be to require that *any* operation producing sets out
5 of sets can be iterated up to | any ordinal number (that is, order type of a well-ordered set). But are we at an end now? No, because we can require

[3]In the axiomatizations of non-mathematical disciplines such as physical geometry, mathematics proper is presupposed; and the axiomatization refers to the content of the discipline under consideration only insofar as it goes beyond mathematics proper. This content, at least in the examples which have been encountered so far, can be expressed by a finite number of axioms.

[4]This circumstance, in the usual presentation of the axioms, is not directly apparent, but shows itself on closer examination of the meaning of the axioms.

[5]The operation "set of" is substantially the same as the operation "power set", where the power set of M is by definition the set of all subsets of M.

[6]In order to carry out the iteration one may put $A = B$ and assume that a special well-ordering has been assigned to any set. For ordinals of the second kind ⟦limit ordinals⟧, the set of the previously obtained sets is always to be formed.

not only that the procedure just described can be carried out with any operation, but that moreover there should exist a set closed with respect to it, that is, one which has the property that, if this procedure (with any operation) is applied to elements of this set, it again yields elements of this set. You will realize, I think, that we are still not at an end, nor can there ever be an end to *this* procedure of forming the axioms, because the very formulation of the axioms up to a certain stage gives rise to the next axiom. It is true that in the mathematics of today the higher levels of this hierarchy are practically never used. It is safe to say that 99.9% of present-day mathematics is contained in the first three levels of this hierarchy. So for all practical purposes, all of mathematics *can* be reduced to a finite number of axioms. However, | this is a mere historical accident, which is of no impor- 6
tance for questions of principle. Moreover it is not altogether unlikely that this character of present-day mathematics may have something to do with another character of it, namely, its inability to prove certain fundamental theorems, such as, for example, Riemann's hypothesis, in spite of many years of effort. For it can be shown that the axioms for sets of high levels, in their relevance, are by no means confined to these sets, but, on the contrary, have consequences even for the 0-level, that is, the theory of integers. To be more exact, each of these set-theoretical axioms entails the solution of certain diophantine problems which had been undecidable on the basis of the preceding axioms.[7] *The diophantine problems in question are of the following type: Let $P(x_1, \ldots, x_n, y_1, \ldots, y_m)$ be a polynomial with given integral coefficients and $n+m$ variables, | $x_1, \ldots x_n, y_1, \ldots, y_m$, and consider* 7
the variables x_i as the unknowns and the variables y_i as parameters; then the problem is: Has the equation $P = 0$ integral solutions for any integral values of the parameters, or are there integral values of the parameters for which this equation has no integral solutions? To each of the set-theoretical axioms a certain polynomial P can be assigned, for which the problem just formulated becomes decidable owing to this axiom. It even can always be achieved that the degree of P is not higher than 4. ⟦The⟧ mathematics of today has not yet learned to make use of the set-theoretical axioms for the solution of number-theoretical problems, except for the axioms of the first level. These are actually used in analytic number theory. But for mastering number theory this is demonstrably insufficient. Some kind of

[7]This theorem, in order to hold also if the intuitionistic or finitistic standpoint is assumed, requires as a hypothesis the consistency of the axioms of set theory, which of course is self-evident (and therefore can be dropped as a hypothesis) if set theory is considered to be mathematics proper. However, for finitistic mathematics a similar theorem holds, without any questionable hypothesis of consistency; namely, the introduction of recursive functions of higher and higher order leads to the solution of more and more number-theoretical problems of the specified kind. In intuitionistic mathematics there doubtless holds a similar theorem for the introduction (by new axioms) of greater and greater ordinals of the second number class.

8 set-theoretical number theory, still to be dis|covered, would certainly reach much farther.

I have tried so far to explain the fact I call ⟦the⟧ incompletability of mathematics for one particular approach to the foundations of mathematics, namely axiomatics of set theory. That, however, this fact is entirely independent of the particular approach and standpoint chosen appears from certain very general theorems. The first of these theorems simply states that, *whatever well-defined system of axioms and rules of inference may be chosen, there always exist diophantine problems of the type described which are undecidable by these axioms and rules, provided only that no false propositions of this type are derivable.*[8] If I speak of a well-defined system of axioms and rules here, this only means that it must be possible actually to write the axioms down in some precise formalism or, if their number is infinite, a finite procedure for writing them down one after the other must be given. Likewise the rules of inference are to be such that, given any premises, either the conclusion (by any one of the rules of inference) can
9 be written | down, or it can be ascertained that there exists no immediate conclusion by the rule of inference under consideration. This requirement for the rules and axioms is equivalent to the requirement that it should be possible to build a finite machine, in the precise sense of a "Turing machine", which will write down all the consequences of the axioms one after the other. For this reason, the theorem under consideration is equivalent to the fact that there exists no finite procedure for the systematic decision of all diophantine problems of the type specified.

The second theorem has to do with the concept of freedom from contradiction. For a well-defined system of axioms and rules the question of their consistency is, of course, itself a well-defined mathematical question. Moreover, since the symbols and propositions of ⟦any⟧ one formalism are always at most enumerable, everything can be mapped on⟦to⟧ the integers, and it is plausible and in fact demonstrable that the question of consistency can always be transformed into a number-theoretical question (to be more
10 exact, into one of the type described above). Now | the theorem says that *for any well-defined system of axioms and rules, in particular, the proposition stating their consistency*[9] *(or rather the equivalent number-theoretical proposition) is undemonstrable from these axioms and rules, provided these axioms and rules are consistent and suffice to derive a certain portion*[10] *of*

[8] This hypothesis can be replaced by consistency (as shown by Rosser in ⟦his *1936*⟧), but the undecidable propositions then have a slightly more complicated structure. Moreover, the hypothesis must be added that the axioms imply the primitive properties of addition, multiplication and $<$.

[9] It is one of the propositions which are undecidable, provided that no false number-theoretical ⟦propositions⟧ are derivable (see the preceding theorem).

[10] Namely, Peano's axioms and the rule of definition by ordinary induction, with a logic satisfying the strictest finitistic requirements.

the finitistic arithmetic of integers. It is *this* theorem which makes the incompletability of mathematics particularly evident. For, *it makes it impossible that someone should set up a certain well-defined system of axioms and rules and consistently make the following assertion about it: All of these axioms and rules I perceive (with mathematical certitude) to be correct, and moreover I believe that they contain all of mathematics.* If someone makes such a statement he contradicts himself.[11] For if he perceives the axioms under consideration to be correct, he also perceives (with the same certainty) that they are consistent. Hence he has a mathematical | 11
insight not derivable from his axioms. However, one has to be careful in order to understand clearly the meaning of this state of affairs. Does it mean that no well-defined system of correct axioms can contain all of mathematics proper? It does, if by mathematics proper is understood the system of all true mathematical propositions; it does not, however, if one understands by it the system of all demonstrable mathematical propositions. I shall distinguish these two meanings of mathematics as mathematics in the objective and in the subjective sense: Evidently no well-defined system of correct axioms can comprise all ⟦of⟧ objective mathematics, since the proposition which states the consistency of the system is true, but not demonstrable in the system. However, as to subjective mathematics, it is not precluded that there should exist a finite rule producing all its evident axioms. However, if such a rule exists, we with our human understanding could certainly never know it to be such, that is, we could never know with mathematical | certainty that all propositions it produces are correct;[12] or 12
in other terms, we could perceive to be true only one proposition after the other, for any finite number of them. The assertion, however, that they are all true could at most be known with empirical certainty, on the basis of a sufficient number of instances or by other inductive inferences.[13] If it were so, this would mean that the human mind (in the realm of pure

[11]If he only says "I believe I shall be able to perceive one after the other to be true" (where their number is supposed to be infinite), he does not contradict himself. (See below.)

[12]For this (or the consequence concerning the consistency of the axioms) would constitute a mathematical insight not derivable from the axioms ⟦and⟧ rules under consideration, contrary to the assumption.

[13]For example, it is conceivable (although far outside the limits of present-day science) that brain physiology would advance so far that it would be known with empirical certainty

1. that the brain suffices for the explanation of all mental phenomena and is a machine in the sense of Turing;
2. that such and such is the precise anatomical structure and physiological functioning of the part of the brain which performs mathematical thinking.

Furthermore, in case the finitistic (or intuitionistic) standpoint is taken, such an inductive inference might be based on a (more or less empirical) belief that non-finitistic (or non-intuitionistic) mathematics is consistent.

mathematics) *is* equivalent to a finite machine that, however, is unable to understand completely[14] its own functioning. This inability ⟦of man⟧ to understand himself would then wrongly appear to him as its ⟦(the mind's)⟧ boundlessness or inexhaustibility. But, please, note that if it were so, this would in no way derogate from the incompletability of objective mathematics. On the contrary, it would only make it particularly striking. For if the human mind were equivalent to a finite machine, then objective mathematics not only would be incompletable in the sense of not being contained in
13 any well-defined axiomatic system, but moreover there would exist | *absolutely* unsolvable diophantine problems of the type described above, where the epithet "absolutely" means that they would be undecidable, not just within some particular axiomatic system, but by *any* mathematical proof the human mind can conceive. So the following disjunctive conclusion is inevitable: *Either mathematics is incompletable in this sense, that its evident axioms can never be comprised in a finite rule, that is to say, the human mind (even within the realm of pure mathematics) infinitely surpasses the powers of any finite machine, or else there exist absolutely unsolvable diophantine problems of the type specified* (where the case that both terms of the disjunction are true is not excluded, so that there are, strictly speaking, three alternatives). It is this mathematically established fact which seems to me of great philosophical interest. Of course, in this connection it is of great importance that at least this fact is entirely independent of the special standpoint taken toward the foundations of mathematics.[15]

14 | There is, however, one restriction to this independence, namely, the standpoint taken must be liberal enough to admit propositions about all integers as meaningful. If someone were so strict a finitist that he would maintain that only particular propositions of the type $2+2=4$ belong to mathematics proper,[16] then the incompletability theorem would not

[14]Of course, the physical working of the thinking mechanism could very well be completely understandable; the insight, however, that this particular mechanism must always lead to correct (or only consistent) results would surpass the powers of human reason.

[15]For intuitionists and finitists the theorem holds as an implication (instead of a disjunction). It is to be noted that intuitionists have always asserted the first term of the disjunction (and negated the second term, in the sense that no demonstrably undecidable propositions can exist). [See above, p. ⟦?⟧[a]]. But this means nothing for the question which alternative applies to intuitionistic mathematics, if the terms occurring in it are understood in the objective sense (rejected as meaningless by the intuitionists). As for finitism, it seems very likely that the first disjunctive term is false.

[16]K. Menger's "implicationistic standpoint" (see *Menger 1930a*, p. 323), if taken in the strictest sense, would lead to such an attitude, since according to it, the only meaningful mathematical propositions (that is, in my terminology, the only ones belonging to mathematics proper) would be those that assert that such and such a conclusion can

---

[a]We are unable to locate a place in the text to which Gödel would be referring here.

apply—at least not *this* incompletability theorem. But I don't think that such an attitude could be maintained consistently, because it is by exactly the same kind of evidence that we judge that $2 + 2 = 4$ and that $a + b = b + a$ for any two integers a, b. Moreover, this standpoint, in order to be consistent, would have to exclude also *concepts* that refer to *all* integers, such as "+" (or to all formulas, such as "correct proof by such and such rules") and replace them with others that apply only within some finite domain of integers (or formulas). It is to be noted, however, that although the truth of the disjunctive theorem is independent of the standpoint taken, the question as to which alternative holds need not be independent of it. (See footnote ⟦15⟧.)

| I think I now have explained sufficiently the mathematical aspect of the 15
situation and can turn to the philosophical implications. Of course, in consequence of the undeveloped state of philosophy in our days, you must not expect these inferences to be drawn with mathematical rigour.

Corresponding to the disjunctive form of the main theorem about the incompletability of mathematics, the philosophical implications *prima facie* will be disjunctive too; however, under either alternative they are very decidedly opposed to materialistic philosophy. Namely, if the first alternative holds, this seems to imply that the working of the human mind cannot be reduced to the working of the brain, which to all appearances is a finite machine with a finite number of parts, namely, the neurons and their connections. So apparently one is driven to take some vitalistic viewpoint. On the other hand, the second alternative, where there exist absolutely undecidable mathematical propositions, seems to disprove the view that mathematics is only our own creation; | for the creator necessarily knows 16
all properties of his creatures, because they can't have any others except those he has given to them. So this alternative seems to imply that mathematical objects and facts (or at least *something* in them) exist objectively and independently of our mental acts and decisions, that is to say, ⟦it seems to imply⟧ some form or other of Platonism or "realism" as to the mathe-

be drawn from such and such axioms and rules of inference in such and such ⟦a⟧ manner. This, however, is a proposition of exactly the same logical character as $2+2 = 4$. Some of the untenable consequences of this standpoint are the following: A negative proposition to the effect that the conclusion B cannot be drawn from the axioms and rules A would not belong to mathematics proper; hence nothing could be known about it except perhaps that it follows from certain other axioms and rules. However, a proof that it does so follow (since these other axioms and rules again are arbitrary) would in no way exclude the possibility that (in spite of the formal proof to the contrary) a derivation of B from A might some day be accomplished. For the same reason also, the usual inductive proof for $a + b = b + a$ would not exclude the possibility of discovering two integers not satisfying this equation.

matical objects.[17] For, the empirical interpretation of mathematics,[18] that
is, the view that mathematical facts are a special kind of physical or psy-
.7 chological facts, is too absurd to be seriously maintained (see below). |
It is not known whether the first alternative holds, but at any rate it is
in good agreement with the opinions of some of the leading men in brain
.8 and nerve physiology, who very decidedly deny the possibility | of a purely
mechanistic explanation of psychical and nervous processes.

As far as the second alternative is concerned, one might object that the constructor need not necessarily know *every* property of what he constructs. For example, we build machines and still cannot predict their behaviour in every detail. But this objection is very poor. For we don't create the machines out of nothing, but build them out of some given material. If the situation were similar in mathematics, then this material or basis for our constructions would be something objective and would force some realistic viewpoint upon us even if certain other ingredients of mathematics were our own creation. The same would be true if in our creations we were to use some instrument in us but different from our ego (such as "reason" interpreted as something like a thinking machine). For mathematical facts would then (at least in part) express properties of this instrument, which would have an objective existence.

One may thirdly object that the meaning of a proposition about all integers, since it is impossible to verify it for all integers one by one, can consist only in the existence of a general proof. Therefore, in the case of an

[17]There exists no term of sufficient generality to express exactly the conclusion drawn here, which only says that the objects and theorems of mathematics are as objective and independent of our free choice and our creative acts as is the physical world. It determines, however, in no way what these objective entities are—in particular, whether they are located in nature or in the human mind or in neither of the two. These three views about the nature of mathematics correspond exactly to the three views about the nature of concepts, which traditionally go by the names of psychologism, Aristotelian conceptualism and Platonism.

[18]That is, the view that mathematical objects and the way in which we know them are not essentially different from physical or psychical objects and laws of nature. The true situation, on the contrary, is that if the objectivity of mathematics is assumed, it follows at once that its objects must be totally different from sensual objects because

1. Mathematical propositions, if properly analyzed, turn out to assert nothing about the actualities of the space-time world. This is particularly clear in applied propositions such as: Either it has or it has not rained yesterday. The existence of purely conceptual knowledge (besides mathematics) satisfying these requirements is not excluded by this remark.
2. The mathematical objects are known precisely, and general laws can be recognized with certainty, that is, by deductive, not inductive, inference.
3. They can be known (in principle) without using the senses (that is, by means of reason alone) for this very reason, that they don't concern actualities about which the senses (the inner sense included) inform us, but possibilities and impossibilities.

undecidable proposition about all integers, neither itself nor its negation is
true. Hence neither expresses an objectively existing but unknown property
of the integers. | I am not in a position now to discuss the epistemological 19
question as to whether this opinion is at all consistent. It certainly looks as if one must *first* understand the meaning of a proposition *before* he can understand a proof of it, so that the meaning of "all" could not be defined in terms of the meaning of "proof". But independently of this epistemological investigation, I wish to point out that one may conjecture the truth of a universal proposition (for example, that I shall be able to verify a certain property for *any* integer given to me) and at the same time conjecture that no general proof for this fact exists. It is easy to imagine situations in which both these conjectures would be very well founded. For the first half of it, this would, for example, be the case if the proposition in question were some equation $F(n) = G(n)$ of two number-theoretical functions which could be verified up to *very* great numbers n.[19] Moreover, exactly as in the natural sciences, this *inductio per enumerationem simplicem* is by no means the only inductive method conceivable in mathematics. I admit that every mathematician has an inborn abhorrence to giving more than
heuristic | significance to such inductive arguments. I think, however, that 20
this is due to the very prejudice that mathematical objects somehow have no real existence. If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics. The fact is that in mathematics we still have the same attitude today that in former times one had toward all science, namely, we try to derive everything by cogent proofs from the definitions (that is, in ontological terminology, from the essences of things). Perhaps this method, if it claims monopoly, is as wrong in mathematics as it was in physics.

This whole consideration incidentally shows that the philosophical implications of the mathematical facts explained do not lie entirely on the side of rationalistic or idealistic philosophy, but that in one respect they
favor the empiricist viewpoint.[20] | It is true that only the second alterna- 21
tive points in this direction. However, ***and this is the item I would like to***

[19] Such a verification of an *equality* (not an inequality) between two number-theoretical functions of not too complicated or artificial structure would certainly give a great probability to their complete equality, although its numerical value could not be estimated in the present state of science. However, it is easy to give examples of general propositions about integers where the probability can be estimated even now. For example, the probability of the proposition which states that for each n there is at least one digit $\neq 0$ between the n-th and n^2-th digits of the decimal expansion of π converges toward 1 as one goes on verifying it for greater and greater n. A similar situation also prevails for Goldbach's and Fermat's theorems ⟦*sic*⟧.

[20] To be more precise, it suggests that the situation in mathematics is not so very different from that in the natural sciences. As to whether, in the last analysis, apriorism or empiricism is correct is a different question.

discuss now, it seems to me that the philosophical conclusions drawn under the second alternative, in particular, conceptual realism (Platonism), are supported by modern developments in the foundations of mathematics also, irrespectively of which alternative holds. The main arguments pointing in this direction seem to me ⟦to be⟧ the following. First of all, if mathematics were our free creation, ignorance as to the objects we created, it is true, might still occur, but only through lack of a clear realization as to what we really have created (or, perhaps, due to the practical difficulty of too complicated computations). Therefore it would have to disappear (at least in principle, although perhaps not in practice[21]) as soon as we attain perfect clearness. However, modern developments in the foundations of mathematics have accomplished an insurmountable degree of exactness, but this has helped practically nothing for the solution of mathematical problems.

22 | Secondly, the activity of the mathematician shows very little of the freedom a creator should enjoy. Even if, for example, the axioms about integers were a free invention, still it must be admitted that the mathematician, after he has imagined the first few properties of his objects, is at an end with his creative ability, and he is not in a position also to create the validity of the theorems at his will. If anything like creation exists at all in mathematics, then what any theorem does is exactly to restrict the freedom of creation. That, however, which restricts it must evidently exist independently of the creation.[22]

Thirdly, if mathematical objects are our creations, then evidently integers and sets of integers will have to be two different creations, the first of which does not necessitate the second. However, in order to prove certain propositions about integers, the concept of set of integers is necessary. So
23 here, in order to find out what properties *we* have | given to certain objects of our imagination, ⟦we⟧ must first create certain other objects—a very strange situation indeed!

[21] That is, every problem would have to be reducible to some finite computation.

[22] It is of no avail to say that these restrictions are brought about by the requirement of consistency, which itself is our free choice, because one might choose to bring about consistency *and* certain theorems. Nor does it help to say that the theorems only repeat (wholly or in part) the properties first invented, because then the exact realization of what was first assumed would have to be sufficient for deciding any question of the theory, which is disproved by the first ⟦argument (above)⟧ and the third argument ⟦(below)⟧. As to the question of whether undecidable propositions can be decided arbitrarily by a new act of creation, see fn. ⟦?⟧[b].

---

[b] No footnote in the manuscript deals with this question. However, the shorthand annotation to p. 29′ (see editorial note g below and the textual notes) does contain the phrase "contin⟦uous⟧ creation". That could have been a note of Gödel to himself to write something on the question.

What I ⟦have⟧ said so far has been formulated in terms of the rather vague concept of "free creation" or "free invention". There exist attempts to give a more precise meaning to this term. However, this only has the consequence that also the disproof of the standpoint in question is becoming more precise and cogent. I would like to show this in detail for the most precise, and at the same time most radical, formulation that has been given so far. It is that which[c] || interprets mathematical propositions as expressing solely certain aspects of syntactical (or linguistic)[23] conventions, that is,

[23]That is, the conventions must not refer to any extralinguistic objects (as does a demonstrat⟦ion⟧-def⟦inition⟧[d]), but must state rules about the meaning or truth of symbolic expressions solely on the basis of their outward structure. Moreover, of course these rules must be such that they do not imply the truth or falsehood of any factual propositions (since in that case they could certainly not be called void of content nor syntactical). This, however, entails their consistency, because an inconsistency (in classical logic, which is under consideration here) would imply every factual proposition. It is to be noted that if the term "syntactical rule" is understood in this generality, the view under consideration includes, as a special elaboration of it, the formalistic foundation of mathematics, since according to the latter, mathematics is based solely on certain syntactical rules of the form: Propositions of such and such structure are true [the axioms], and if propositions of ... structure are true, then such and such other propositions are also true; and moreover, as can easily be seen, the consistency proof gives the assurance that these rules are void of content insofar as they imply no factual propositions. On the other hand, also, vice-versa, it will turn out below that the feasibility of the nominalistic program implies the feasibility of the formalistic program. (For very lucid expositions of the philosophical aspects of this nominalistic view, see *Hahn 1935* or *Carnap 1935a*, *1935b*.) It might be doubted whether this (nominalistic) view should at all be subsumed under the view that considers mathematics to be a free creation of the mind, because it denies altogether the existence of mathematical objects. Moreover, the relationship between the two is extremely close, since also under the other view the so-called existence of mathematical objects consists solely in their being constructed in thought, and nominalists would not deny that we actually imagine (non-existent) objects behind the mathematical symbols and that these subjective ideas might even furnish the guiding principle in the choice of the syntactical rules.

[c]The double vertical lines indicate material marked in the manuscript "Omit from here to p. 29". Since this material was not crossed out, a plausible conjecture is that it was to be omitted only from his oral presentation. But other conjectures are possible, for example, that he came at a later time to think it duplicative of or superseded by discussions in **1953/9*. See also the textual notes.

[d]Gödel writes "demonstrat.-def.", in some other places without the hyphen. It might with almost equal plausibility be read as "demonstrative definition" (which would be stylistically more attractive).

What indication there is as to what he has in mind is given by the following passage:

> Of course it is to be noted that a demonstrat⟦ion⟧-def⟦inition⟧ does not mean pointing the finger to the object for which a name is introduced (which in most cases is not possible even for physical concepts), but that it rather means explaining the meaning of a word by means of the situations in which it is used.

(From Gödel's footnote 58. This note is flagged in the alternate version of the text printed from p. 29′ [see the textual notes] and also in Gödel's footnote 26, to which we have found no reference in the text.)

they simply repeat parts of these conventions. According to this view,
mathematical propositions, duly analyzed, must turn out to be as void
of content as, for example, the statement "All stallions are horses". Ev-
erybody will agree that this proposition does not express any zoological
or other objective fact, but ⟦rather,⟧ its truth is due solely to the circum-
stance that we chose to use the term "stallion" as an abbreviation for "male
24 horse". | Now by far the most common type of symbolic conventions are
definitions (either explicit or contextual, where the latter however must be
such as to make it possible to eliminate the term defined in any context
⟦where⟧ it occurs). Therefore the simplest version of the view in ques-
tion would consist in the assertion that mathematical propositions are true
solely owing to the definitions of the terms occurring in them, that is, that
by successively replacing all terms by their definientia, any theorem can be
reduced to an explicit tautology, $a = a$. (Note that $a = a$ must be admitted
as true if definitions are admitted, for one may define b by $b = a$ and then,
owing to this definition, replace b by a in this equality.) But now it follows
directly from the theorems mentioned before that such a reduction to ex-
plicit tautologies is impossible. For it would immediately yield a mechanical
procedure for deciding about the truth or falsehood of every mathematical
proposition. Such a procedure, however, cannot exist, not even for number
25 theory. This disproof, it is true, refers only to the simplest | version of this
(nominalistic) standpoint. But the more refined ones do not fare any bet-
ter. The weakest statement that at least would have to be demonstrable, in
order that this view concerning the tautological character of mathematics
be tenable, is the following: Every demonstrable mathematical proposition
can be deduced from the rules about the truth and falsehood of sentences
alone (that is, without using or knowing anything else except these rules)
and the negations of demonstrable mathematical propositions cannot be
so derived.[24] In precisely formulated languages, such rules (that is, rules
which stipulate under which conditions a given sentence is true) occur as
a means for determining the meaning of sentences. Moreover in all known
languages there *are* propositions which seem to be true owing to these rules
alone. For example, if disjunction and negation are introduced by those
rules:

1) $p \vee q$ is true if at least one of its terms is true, and
2) $\sim p$ is true if p is not true,

[24]As to the requirement of consistency, see fn. ⟦23?⟧[e].

[e]It is also possible that Gödel intended to write a new note on this subject. In the manuscript, the text as we give it is above some crossed-out text in which something is said about the "requirement of consistency", which, however, he may have thought repeated points in (our) note 23.

then it clearly follows from these rules | that $p \vee \sim p$ is always true what- 26
ever p may be. (Propositions so derivable are called tautologies.) Now it is actually so, that for the symbolisms of mathematical logic, with suitably chosen semantical rules, the truth of the mathematical axioms *is* derivable from these rules;[25] however (and this is the great stumbling block), in this derivation the mathematical and logical concepts and axioms themselves must be used in a special application, namely, as referring to symbols, combinations of symbols, sets of such combinations, etc. Hence this theory, if it wants to prove the tautological character of the mathematical axioms, must first assume these axioms to be true. So while the original idea of this viewpoint was to make the truth of the mathematical axioms understandable by showing that they are tautologies, it ends up with just the opposite, that is, the truth of the axioms must *first* be assumed and *then*
it can be shown that, in a suitably chosen language, | they are tautologies. 27
Moreover, a similar statement holds good for the mathematical concepts, that is, instead of being able to define their meaning by means of symbolic conventions, one must first know their meaning in order to understand the syntactical conventions in question or the proof that they imply the mathematical axioms but not their negations. Now, of course, it is clear that this elaboration of the nominalistic view does not satisfy the requirement set up on page ⟦25?⟧, because not the syntactic rules alone, but all of mathematics in addition is used in the derivations. But moreover, this elaboration of nominalism would yield an outright *disproof* of it (I must confess I can't picture any better disproof of this view than this proof of it), provided that one thing could be added, namely, that the outcome described is unavoidable (that is, independent of the particular symbolic language and interpretation of mathematics chosen). Now it is not exactly this that can be proved, but something so close to it that it also suffices to disprove the view in question. Namely, it follows by the metatheorems mentioned that a proof for the tautological character (in a suitable language) of the mathematical axioms is at the same time a proof for their consistency, and cannot be achieved with any *weaker* means of proof than are contained in these axioms themselves. This does not mean that *all* the axioms of a given system must be used in its consistency proof. On the contrary, usually the axioms lying outside the system which are necessary make it possible to

[25]See *Ramsey 1926*, pp. 368 and 382, and *Carnap 1937*, pp. 39 and 110. It is worth mentioning that Ramsey even succeeds in reducing them to explicit tautologies $a = a$ by means of explicit definitions (see p. ⟦24?⟧ above), but at the expense of admitting propositions of infinite (and even transfinite) length, which of course entails the necessity of presupposing transfinite set theory in order to be able ⟦to⟧ deal with these infinite entities. Carnap confines himself to propositions of finite length, but instead has to consider infinite sets, sets of sets, etc., of these finite propositions.

dispense with some of the axioms of the system (although they do not imply the latter).[26] However, what follows with practical certainty is this: In order to prove the consistency of classical number theory (and *a fortiori* of all stronger systems) certain *abstract* concepts (and the directly evident axioms referring to them) must be used, where "abstract" means concepts which do not refer to sense objects,[27] of which symbols are a special kind. These abstract concepts, however, are certainly not syntactical [but rather those whose justification by syntactical considerations should be the main task of nominalism]. Hence it follows that *there exists no rational justification of our precritical beliefs concerning the applicability and consistency of classical mathematics (nor even its undermost level, number theory) on the basis of a syntactical interpretation.* It is true that this statement does not apply to certain subsystems of classical mathematics, which may even contain some *part* of the theory of the abstract concepts referred to. In this sense, nominalism can point to some partial successes. For it *is* actually possible to base the axioms of these systems on purely syntactical considerations. In this manner, for example, the use of the concepts of "all" and "there is" referring to integers can be justified (that is, proved consistent) by means of syntactical considerations. However, for the most essential number-theoretic axiom, complete induction, such a syntactical foundation, even within the limits in which it is possible, gives no justification of our precritical belief in it, since this axiom itself has to be used

[26]For example, any axiom system S for set theory belonging to the series explained in the beginning of this lecture, the axiom of choice included, can be proved consistent by means of the axiom of the next order (or by means of the axiom that S is consistent) without the axiom of choice. Similarly, it is not impossible that the axioms of the lower levels of this hierarchy could be proved consistent by means of axioms of higher levels, with such restrictions, however, as would make them acceptable to intuitionists.

[27]Examples of such abstract concepts are, for example, "set", "function of integers", "demonstrable" (the latter in the non-formalistic sense of "knowable to be true"), "derivable", etc., or finally "there is", referring to all *possible* combinations of symbols. The necessity of such concepts for the consistency proof of classical mathematics results from the fact that symbols can be mapped on⟦to⟧ the integers, and therefore finitistic (and *a fortiori*, classical) number theory contains all proofs based solely upon them. The evidence for this fact so far is not absolutely conclusive because the evident axioms referring to the non-abstract concept under consideration have not been investigated thoroughly enough. However, the fact itself is acknowledged even by leading formalists; see ⟦*Bernays 1941a*, pp. 144, 147; *1935*, pp. 68, 69; *1935b*, p. 94; *1954*, p. 2; also *Gentzen 1937*, p. 203⟧[f].

[f]These references are supplied from **1953/9-III*, fn. 24, which is attached to essentially the same remark (§24 of that text). But cf. also *1958*, fn. 1.

in the syntactical considerations.[28] The fact that the more modest you are
in the axioms for which you want to set up a tautological interpretation,
the less of mathematics you need in order to do it, has the consequence
that if finally you become so modest as to | confine yourself to some finite 28
domain, for example, to the integers up to 1000, then the mathematical
propositions valid in this field can be so interpreted as to be tautological
even in the strictest sense, that is, reducible to explicit tautologies by means
of the explicit definitions of the terms. No wonder, because the section
of mathematics necessary for the proof of the consistency of this finite
mathematics is contained already in the theory of the finite combinatorial
processes which are necessary in order to reduce a formula to an explicit
tautology by substitutions. This explains the well-known, but misleading,
fact that formulas like $5+7=12$ can, by means of certain definitions, | be 29
reduced to explicit tautologies. This fact, incidentally, is misleading also for
the reason that in these reductions (if they are to be interpreted as simple
substitutions of the definiens for the definiendum on the basis of explicit
definitions), the $+$ is not identical with the ordinary $+$, because it can be
defined only for a finite number of arguments (by an enumeration of this
finite number of cases). If, on the other hand, $+$ is defined contextually,
then one has to use the concept of finite manifold already in the proof of
$2+2=4$. A similar circularity also occurs in the proof that $p \vee \sim p$ is a
tautology, because disjunction and negation, in their intuitive meanings,
evidently occur in it. ||

The essence of this view is that there exists no such thing as a mathematical fact, that the truth of propositions which we believe express mathematical facts only means that (due to the rather complicated rules which define the meaning of propositions, that is, which determine under what circumstances a given proposition is true) an idle running of language occurs in these propositions, in that the said rules make them true no matter what the facts are. Such propositions can rightly be called void of content. Now it ⟦is⟧ actually possible to build up a language in which mathematical propositions are void of content in this sense. The only trouble is

1. that one has to use the very same mathematical facts[g] (or equally

[28]The objection raised here against a syntactical foundation of number theory is substantially the same ⟦as the one⟧ which Poincaré leveled against both Frege's and Hilbert's foundation of number theory. However, this objection is not justified against Frege, because the logical concepts and axioms he has to presuppose do not explicitly contain the concept of a "finite manifold" with its axioms, while the grammatical concepts and considerations necessary to set up the syntactical rules and establish their tautological character do.

[g]An unnumbered remark cited at this point appears at the bottom of page 29′ of Gödel's manuscript text. Neither a true footnote nor a textual insertion, it is rather a shorthand annotation. For a transcription and translation of its contents, see the textual notes.

complicated other mathematical facts) in order to show that they don't exist;

2. that by this method, if a division of the empirical facts in⟦to⟧ two parts, A and B, is given such that B implies nothing in A, a language can be constructed in which the propositions expressing B would be void of content. And if your opponent were to say: "You are arbitrarily disregarding certain observable facts B", one may answer: "You are doing the same thing, for example with the law of complete induction, which I perceive to be true on the basis of my understanding (that is, perception) of the concept of integer."

30 | However, it seems to me that nevertheless one ingredient of this wrong theory of mathematical truth is perfectly correct and really discloses the true nature of mathematics. Namely, it is correct that a mathematical proposition says nothing about the physical or psychical reality existing in space and time, because it is true already owing to the meaning of the terms occurring in it, irrespectively of the world of real things. What is wrong, however, is that the meaning of the terms (that is, the concepts they denote) is asserted to be something man-made and consisting merely in semantical conventions. The truth, I believe, is that these concepts form an objective reality of their own, which we cannot create or change, but only perceive and describe.[29]

Therefore a mathematical proposition, although it does not say anything about space-time reality, still may have a very sound objective content, in-
31 sofar as it | says something about relations of concepts. The existence of non-"tautological" relations between the concepts of mathematics appears

[29]This holds good also for those parts of mathematics which *can* be reduced to syntactic rules (see above). For these rules are based on the idea of a finite manifold (namely, of a finite sequence of symbols), and this idea and its properties are entirely independent of our free choice. In fact, its theory is equivalent to the theory of ⟦the⟧ integers. The possibility of so constructing a language that this theory is incorporated into it in the form of syntactic rules proves nothing. See fn. ⟦?⟧[h].

---

[h]A conjecture as to what Gödel is referring to is that it is his footnote 35, to which there is no reference in the text. It reads as follows:

To be more exact the true situation as opposed to the view criticized is the following:

1. The meanings of mathematical terms are not reducible to the linguistic rules about their use except for a very restricted domain of mathematics (cf. ⟦pp. 25–27?⟧).
2. Even where such a reduction is possible the linguistic rules cannot be considered to be something man-made and propositions about them to be lacking objective content because these rules are based on the idea of a finite manifold (in the form of finite sequences of symbols) and this idea (with all its properties) is entirely independent of any convention and free choice (hence *is* something objective). In fact, its theory is equivalent to arithmetic.

It could be, however, that this note was superseded by our note 29 (Gödel's 49).

above all in the circumstance that for the primitive terms of mathematics, axioms must be assumed, which are by no means tautologies (in the sense of being in any way reducible to $a = a$), but still do follow from the meaning of the primitive terms under consideration. For example, the basic axiom, or rather, axiom schema, for the concept of set of integers says that, given a well-defined property of integers (that is, a propositional expression $\varphi(n)$ with an integer variable n), there exists the set M of those integers which have the property φ. Now, considering the circumstance that φ may itself contain the term "set of integers", we have here a series of rather involved
axioms about the concept of set. | Nevertheless, these axioms (as the 32
aforementioned results show) cannot be reduced to anything substantially simpler, let alone to explicit tautologies. It is true that these axioms are valid owing to the meaning of the term "set"—one might even say they express the very meaning of the term "set"—and therefore they might fittingly be called analytic; however, the term "tautological", that is, devoid of content, for them is entirely out of place, because even the assertion of the existence of a concept of set satisfying these axioms (or of the consistency of these axioms) is so far from being empty that it cannot be proved without again using the concept of set itself, or some other abstract concept of ⟦a⟧ similar nature.

Of course, this particular argument is addressed only to mathematicians who admit the general concept of set in mathematics proper. For finitists, however, literally the same argument could be alleged for the concept of integer and the axiom of complete induction. For, if the general concept of
set is *not* admitted in mathematics proper, then complete induction | must 33
be assumed as an axiom.

I wish to repeat that "analytic" here does not mean "true owing to our definitions", but rather "true owing to the nature of the concepts occurring ⟦therein⟧", in contradistinction to "true owing to the properties and the behaviour of things". This concept of analytic is so far from meaning "void of content" that it is perfectly possible that an analytic proposition might
be undecidable (or decidable only with ⟦a certain⟧ probability). | For, our 34
knowledge of the world of concepts may be as limited and incomplete as that of ⟦the⟧ world of things. It is certainly undeniable that this knowledge, in certain cases, not only is incomplete, but even indistinct. This occurs in the paradoxes of set theory, which are frequently alleged as a disproof of Platonism, but, I think, quite unjustly. Our visual perceptions sometimes contradict our tactile perceptions, for example, in the case of a rod immersed in water, but nobody in his right mind will conclude from this fact that the outer world does not exist.

I have purposely spoken of two separate worlds (the world of things and of concepts), because I do not think that Aristotelian realism (according to which concepts are parts or aspects of things) is tenable.

| Of course I do not claim that the foregoing considerations amount to 35

a real proof of this view about the nature of mathematics. The most I could assert would be to have disproved the nominalistic view, which considers mathematics to consist solely in syntactical conventions and their consequences. Moreover, I have adduced some strong arguments against the more general view that mathematics is our own creation. There are, however, other alternatives to Platonism, in particular psychologism and Aristotelian realism. In order to establish Platonistic realism, these theories would have to be disproved one after the other, and then it would have to be shown that they exhaust all possibilities. I am not in a position to do this now; however, I would like to give some indications along these lines. One possible form of psychologism admits that mathematics investigates relations of concepts and that concepts cannot be created at our will, but
36 are given to us as a reality, which | we cannot change; however, it contends that these concepts are only psychological dispositions, that is, that they are nothing but, so to speak, the wheels of our thinking machine. To be more exact, a concept would consist in the disposition

1. to have a certain mental experience when we think of it

and

2. to pass certain judgements (or have certain experiences of direct knowledge) about its relations to other concepts and to empirical objects.

The essence of this psychologistic view is that the object of mathematics is nothing but the psychological laws by which thoughts, convictions, and so on occur in us, in the same sense as the object of another part of psychology is the laws by which emotions occur in us. The chief objection to this view I can see at the present moment is that if it were correct, we would have no mathematical knowledge whatsoever. We would not know, for example, that $2 + 2 = 4$, but only that our mind is so constituted as to hold this to be true, and there would then be no reason whatsoever why, by some
37 other train of thought, we should not arrive at the opposite | conclusion with the same degree of certainty. Hence, whoever assumes that there is some domain, however small, of *mathematical* propositions which we *know* to be true, cannot accept this view.[i]

38 | I am under the impression that after sufficient clarification of the concepts in question it will be possible to conduct these discussions with mathematical rigour and that the result then will be that (under certain assumptions which can hardly be denied [in particular the assumption that there exists at all something like mathematical knowledge]) the Platonistic view

[i]The rest of this page and part of the next are crossed out in Gödel's manuscript.

is the only one tenable. Thereby I mean the view that mathematics describes a non-sensual reality, which exists independently both of the acts and ⟦of⟧ the dispositions of the human mind and is only perceived, and probably perceived very incompletely, by the human mind. This view is rather unpopular among mathematicians; there exist, however, some great mathematicians who have adhered to it. For example, Hermite once wrote the following sentence:

> Il existe, si je ne me trompe, tout un monde qui est l'ensemble 39
> des vérités mathématiques, dans lequel nous n'avons accès que par l'intelligence, comme existe le monde des réalités physiques; l'un et l'autre indépendants de nous, tous deux de création divine.[30] ⟦There exists, unless I am mistaken, an entire world consisting of the totality of mathematical truths, which is accessible to us only through our intelligence, just as there exists the world of physical realities; each one is independent of us, both of them divinely created.⟧

[30]See *Darboux 1912*⟦, p. 142⟧. The passage quoted continues as follows:

> qui ne semblent distincts qu'à cause de la faiblesse de notre esprit, qui ne sont pour une pensée plus puissante qu'une seule et même chose, et dont la synthèse se révèle partiellement dans cette merveilleuse correspondance entre les Mathématiques abstraites d'une part, l'Astronomie et toutes les branches de la Physique de l'autre. ⟦and appear different only because of the weakness of our mind; but, for a more powerful intelligence, they are one and the same thing, whose synthesis is partially revealed in that marvelous correspondence between abstract mathematics on the one hand and astronomy and all branches of physics on the other.⟧

So here Hermite seems to turn toward Aristotelian realism. However, he does so only figuratively, since Platonism remains the only conception understandable for the human mind.